Lagrange Multiplier (Part III)

Liming Pang

In this Part we will do more examples of the method of Lagrange multiplier.

Example 1. *If the utility function is $u(x,y)$ and the constraint is the budget $g(x,y) = px + qy = m$, in order to maximize the utility under the given constraint m, we need*

$$\begin{cases} \nabla u(x,y) = \lambda g(x,y) \\ px + qy = m \end{cases}$$

The first condition implies

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = \lambda p \\ \frac{\partial u}{\partial y}(x,y) = \lambda q \end{cases}$$

We thus get

$$\frac{\frac{\partial u}{\partial x}(x,y)}{\frac{\partial u}{\partial y}(x,y)} = \frac{p}{q}$$

at the maximal point. Note that the left hand side of the above equation is the marginal rate of substitution (MRS) of the two goods. This result tells us in order to maximize the utility, we should find the point on the budget that the marginal rate of substitution equals to the ration of their prices.

Example 2. *Find the shortest distance between the hyperbola $y = \frac{1}{x}$ and the origin*

If (x,y) is a point on the plane, its distance to $(0,0)$ is $d = \sqrt{x^2+y^2}$. To minimize d, we can minimize $f(x,y) = d^2 = x^2 + y^2$ instead. Since we only care about points on the hyperbola $y = \frac{1}{x}$, the constraint is $g(x,y) = xy = 1$.

$$\begin{cases} \nabla f(x,y) = \lambda g(x,y) \\ xy = 1 \end{cases}$$

i.e.

$$\begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 1 \end{cases}$$

We get $\lambda = 2, x = 1, y = 1, f(1,1) = 2$, *or* $\lambda = 2, x = -1, y = -1, f(1,1) = 2$. *So the points closest to origin are* $(1,1)$ *and* $(-1,-1)$, *the distance is* $d = \sqrt{2}$

Example 3. *Three alleles (alternative versions of a gene) A,B,O determine the four blood types A(AA or AO), B(BB or BO), O(OO) and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is*

$$P = 2pq + 2pr + 2rq$$

where p, q, r *are the proportions of* A, B, O *in the population. Use the fact* $p + q + r = 1$ *to show that* P *is at most* $\frac{2}{3}$.

We will maximize $P = 2pq + 2pr + 2rq$, *with constraint* $g(p.q.r) = p + q + r = 1$.

$$\begin{cases} \nabla P = \lambda \nabla g \\ p + q + r = 1 \end{cases}$$

i.e.

$$\begin{cases} 2(q + r) = \lambda \\ 2(p + r) = \lambda \\ 2(p + q) = \lambda \\ p + q + r = 1 \end{cases}$$

The solution is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, *with* $P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{2}{3}$. *The way to test whether* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *is a maximum or minimum is to compare with any other point on the constraint, say* $(1,0,0)$: $P(1,0,0) = 0 < f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, *so* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *cannot be minimum, it has to be maximum. We conclude* $P \leq \frac{2}{3}$.

Example 4. *If a triangle has three edges of length a, b, c, by Heron's Formula, its area is given by*

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is half of the perimeter.

If it is known the perimeter of a triangle is $2s$, what is the largest possible area of the triangle?

We will maximize the square of the area of the triangle, which is

$$f(a,b,c) = s(s-a)(s-b)(s-c)$$

under the constraint $g(a,b,c) = a+b+c = 2s$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(a,b,c) = 2s \end{cases}$$

which are indeed

$$\begin{cases} -s(s-b)(s-c) = \lambda \\ -s(s-a)(s-c) = \lambda \\ -s(s-a)(s-b) = \lambda \\ a+b+c = 2s \end{cases}$$

The solution is $a = b = c = \frac{2s}{3}$, so the largest possible area is

$$\sqrt{f(\frac{2s}{3}, \frac{2s}{3}, \frac{2s}{3})} = \frac{s^2}{3\sqrt{3}} = \frac{s^2}{9}\sqrt{3}$$

Example 5. *Find the largest and smallest volumes of a rectangular box whose surface area is 1500 and whose total edge length is 200*

We will find the maximum and minimum of $V = xyz$ with the constraints $2(xy+xz+yz) = 1500$ and $4(x+y+z) = 200$, i.e. $g(x,y,z) = xy+xz+yz = 750$ and $h(x,y,z) = x+y+z = 50$.

So we solve the following system of equations:

$$\begin{cases} \nabla V = \lambda \nabla g + \mu \nabla h \\ xy + xz + yz = 750 \\ x + y + z = 50 \end{cases}$$

i.e.

$$\begin{cases} yz = \lambda(y+z) + \mu \\ xz = \lambda(x+z) + \mu \\ xy = \lambda(x+y) + \mu \\ xy + xz + yz = 750 \\ x + y + z = 50 \end{cases}$$

From the first three equations we get:

$$\begin{cases} (y-x)z = \lambda(y-x) \\ (z-y)x = \lambda(z-y) \\ (x-z)y = \lambda(x-z) \end{cases}$$

Suppose x, y, z *are three different numbers, we get* $x = y = z = \lambda$*, contradiction. So if* (x, y, z) *is a solution, at least two of* x, y, z *are equal. Since* x, y, z *are symmetric, we may assume* x *and* y *are the edges of equal length, i.e.* $x = y$*. Then putting it into*

$$\begin{cases} xy + xz + yz = 750 \\ x + y + z = 50 \end{cases}$$

We get the solutions are $(\frac{50-5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}, \frac{50+10\sqrt{10}}{3})$ *and* $(\frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50-10\sqrt{10}}{3})$*. The volume for the former case is* $\frac{2500}{27}(35-\sqrt{10})$*, and the volume of the later is* $\frac{2500}{27}(35+\sqrt{10})$*. So the maximum of the volume is* $\frac{2500}{27}(35+\sqrt{10})$ *and the minimum of the volume is* $\frac{2500}{27}(35-\sqrt{10})$*.*